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Study of the output of a quantum circuit consisting of four qubits for one input type

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Abstract
We studied the outputs that a quantum circuit consisting of four qubits $|q_0q_1q_2q_3\rangle$ can produce by using quantum gates, which operate on the states of the qubit as a state vector that can be expressed through unitary matrices. We made the change by exploiting the superposition property of the Hadamard gate, and with its presence in the quantum circuit, we used CNOT gates, which produced entangled states . The study explained the output of a quantum circuit consisting of four qubits and gave a methodology for changing this output by installing one input type for the quantum circuit (we used the input $|0000\rangle$). It showed the importance of measurement in the quantum circuit and its effect on the $|input\ state\rangle$ and the $= |output\ state\rangle$. Starting from, produce all 16 states of the four qubits, which can be expressed as the computational basis state of a vector state using only an x-gate.

Keywords: Qubit states, quantum gates, quantum circuit, input and output

Introduction

Quantum computing (QC): is a type of computing that relies on theories of quantum mechanics based on the interpretation of phenomena in terms of probability and the physical world which is inherently unpredictable. Devices that use this type of computing are known as "quantum computers" in which atomic particles such as electrons, photons, ions, etc. are used, unlike the classic circuit in which transistors, logic gates, etc. are used. Subatomic particles as well as the spin and states that can be shared in the system as a whole will work efficiently using memory, which will work more powerfully. So far, no model disobeys Church-Turing theory other than quantum computing, where quantum computers run much faster than classical computers [1]. We may understand that qubits can exist between two states $|0\rangle$ and $|1\rangle$, so it gives more solutions to problems, so we must understand, when the number of qubits increases, the quantum computing power becomes much larger. In other words, the superposition effect is evident when going from a 1-qubit system to an n-qubit system. For example, the chessboard, which explains how powerful exponential growth is. We note that for each square that follows there are multiples of the square before it, and for each subsequent square of the 64 squares we may find the matter simple, but surprisingly, the multipliers gradually increase in the first squares, and we will find a large inflation in the squares that follow until we get almost infinite numbers. The increase follows the formula $2^{(n-1)}$ in square n, because in the 21st square we get a number that exceeds a million. And by extrapolating this here, the odds of finding a solution increase with the number of qubits. For a 3-qubit system, there is a probability of 23, and similarly for a 4-qubit and a 6-qubit system. We find that there are 2300 states of a 300-bit system. Therefore, we discover that the increase in computing speed follows the slight increase in the number of qubits, and this leads to great efficiency and an increase in the speed of information processing. In other words, one quantum bit is equivalent to 2-bits of information, and the computational power increases with the increase in qubits [2]. Quantum computing depends in its tasks and applications on what a quantum computer and quantum information processing can implement. It is useful to mention the architecture of a quantum computer and the applications of quantum computing, Quantum computing applications are used in many areas such as Physics modeling, Chemical reaction modeling, Genetics analysis, Signal processing, Logistics, Weather modeling, Cyber security, Finance, Image processing, Drug discovery [3].

The architecture of a quantum computer can be defined as a hybrid of classical and quantum components consisting of five layers; Application Layer, Classical Layer, Digital Layer, Analog Layer, Quantum Layer and Quantum Processing Unit (QPU) [1]. In our thesis that talks about four qubits, we found there are several studies and research studies that dealt with four qubits and their states. We mention the following: In 2022, a paper was presented by Diana Jingle and others on the design and analysis of the Circuit for Grover's Quantum Search Algorithm on 2, 3 and 4-qubit systems [4], In the same year an innovation was introduced by Sara Giordano and Miguel A. Martin-Delgado through an artificial intelligence algorithm with machine reinforcement learning (Q-learning) to build exquisite entangled states with four qubits [5]. In 2023, Meng-Li Guo and others presented a paper titled Tetrahedron genuine entanglement measurement of four-qubit systems that studies the quantum measurement of true multi-qubit entanglement for four-qubit systems [6]. Ahmad Salmanogli presented an article talking about entangled state engineering in " the 4-Coupled Qubits System" [7].

The aim is to study a relationship that describes what is happening in the quantum circuit in terms of calculations and results by adjusting the quantum gates appropriate for the inputs and suitable for any change to give results that represent outputs that carry information related to the inputs, and with measurement, the outputs will represent possible results for the inputs.

Computation Part

In classical physics, the state of a particle is described through its position and momentum if it is in motion, As we will mention in the postulates. In quantum mechanics, depending on the physical state, the mathematical description of the state of the system takes place on two different levels [8]. Particularly in quantum computing and the operations of quantum devices to represent qubits, we use complex Finite dimensions Hilbert space. The computational basis of $\mathbb{R}^n$, denoted by $\{|0\rangle, \ldots, |n-1\rangle\}$, is given by [8].

$$|\mathbf{0}\rangle = \begin{bmatrix}\mathbf{1}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{bmatrix}, \ldots\ , |\boldsymbol{n}-\mathbf{1}\rangle = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \vdots\\ \mathbf{1}\end{bmatrix}$$

Quantum notation for the basis of a given quantum system which usually refers:

$$|\mathbf{0}\rangle = \begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix} \quad , \quad |\mathbf{1}\rangle = \begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}$$

In quantum information, it must have orthogonal and normalized properties [10]: $\langle \mathbf{0}|\mathbf{0}\rangle = \langle \mathbf{1}|\mathbf{1}\rangle = \mathbf{1}\ , \langle \mathbf{0}|\mathbf{1}\rangle = \langle \mathbf{1}|\mathbf{0}\rangle = \mathbf{1}$
The spin states $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ form a standard orthonormal basis for the (single) qubit's space. So it will be arbitrary state vector as.

$$|\boldsymbol{\psi}\rangle\ = \boldsymbol{\alpha}|\mathbf{0}\rangle\ + \boldsymbol{\beta}|\mathbf{1}\rangle$$

Probability in quantum mechanics the norm of a qubit is equal to 1, Which means $\langle \boldsymbol{\psi}|\boldsymbol{\psi}\rangle = \mathbf{1}$, $|\boldsymbol{\alpha}|^{2}+|\boldsymbol{\beta}|^{2} = \mathbf{1}$. So a qubit is a quantum state vector (a function of the binary degree of freedom 0, 1) which represents a superposition of the states of a single bit. Superposition states are forbidden in a classical computer [11]. In any Hilbert space the quantum state is a vector in $(\mathbb{R}^{2})^{\otimes n}$ [12]. So we can describe the possible string of bits by the computational basis, By describing a state of multiple qubits as a superposition state. For example, The pure state of a quantum is 2-qubit, $\mathbb{R}^{2} \otimes \mathbb{R}^{2} = \mathbb{R}^{4}$.

$$|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle = \begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix} \otimes \begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{1}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix} = |\mathbf{00}\rangle, |\mathbf{0}\rangle \otimes |\mathbf{1}\rangle = \begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix} \otimes \begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{1}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix} = |\mathbf{01}\rangle$$

$$|\mathbf{1}\rangle \otimes |\mathbf{0}\rangle = \begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix} \otimes \begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{1}\\ \mathbf{0}\end{bmatrix} = |\mathbf{10}\rangle, |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle = \begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix} \otimes \begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{1}\end{bmatrix} = |\mathbf{11}\rangle$$

We can write.

$$|\boldsymbol{\psi}\rangle\ = \boldsymbol{a}_{0}|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle + \boldsymbol{a}_{1}|\mathbf{0}\rangle \otimes |\mathbf{1}\rangle + \boldsymbol{a}_{2}|\mathbf{1}\rangle \otimes |\mathbf{0}\rangle + \boldsymbol{a}_{3}|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle$$

Where,

$$|\boldsymbol{a}_{0}|^{2} + |\boldsymbol{a}_{1}|^{2} + |\boldsymbol{a}_{2}|^{2} + |\boldsymbol{a}_{3}|^{2}=1,$$

At one time a register contains many different bits, with different amplitudes as well. This is the general state 2-qubit [13]." now, let's try, the pure state of quantum is 4-qubits.

$$\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{R}^{16}$$

$$|\psi\rangle = a_0|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle + a_1|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle + a_2|0\rangle \otimes |0\rangle \otimes |1\rangle$$

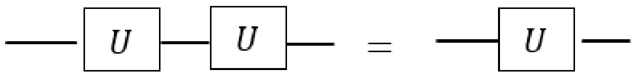
$$\otimes |0\rangle + \cdots\cdots + a_{15}|1\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle$$

Where $|a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots + |a_{15}|^2$=1. The general formula for the pure state of an n-qubit quantum;

$$|\psi\rangle = a_0|00\ldots0\rangle + a_1|00\ldots1\rangle + \cdots + a_{2^n-1}|11\ldots1\rangle = \sum_{i=0}^{2^n-1} a_i\,|i\rangle$$

The above equation represents “computational basis eigenstate” [13].

Considering that the quantum circuit is a quantum system, it has an equation that includes the theory of quantum computing, and what determines this equation in the quantum circuit are the quantum gates, which can deal with the $|states\rangle$ of qubits in the form of assembly,



Because the equation can be expressed by at least one gate, and even if there is no gate, it can be expressed by the I-gate [14]. The identity gate may be included with them as the first Pauli gate [15], but there are three Pauli gates and they are considered the simplest and most important type of gate that is applied to a single qubit, and it has many quantum algorithms [8]. The first Pauli gates have more than one name: the bit-flip gate [16], the quantum NOT- gate, or X gate [15]. the logical basis states **X**-gate;

$$\{|0\rangle \to |1\rangle, |1\rangle \to |0\rangle\}$$

Dirac representation of the **X**-gate [8].

$\{\, X = |1\rangle\langle 0| + |0\rangle\langle 1| \,\}$ the matrix representation for the X-gate: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Table 1: Shows the inputs and outputs of the NOT gate [17]

Input	output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle + \|0\rangle)$
$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|0\rangle)$

To change the state of a qubit, the first thing one comes to think of is an X-gate, Because it can be defined as a state change gate in the "z-basis" located on the Bloch sphere. As it has many mathematical forms, including expressing it with the "x —basis" [18];

$$\mathrm{X-gate} = \textstyle\sum_{x\in\pm} x P_g^x = |+\rangle\langle +| - |-\rangle\langle -|,\ P_g^x = |x_g\rangle\langle x_g|$$

For four qubits $|q_0q_1q_2q_3\rangle$, all states can be generated on a computational basis by changing the $|input\ state\rangle$, Through the *X*-gate and the *I*-gate, if we consider the quantum circuit consisting of four qubits, It consists of two circuit, each circuit consisting of two qubits $|q_iq_i\rangle$,{*X*-gate$|q_i\rangle$, Without a gate, i.e. an *I*-gate$|q_i\rangle$ } [19]. So the gate of two qubits can be expressed by an operator $\hat{U} = XI$,Through matrix representation, computational basis states for two qubit:

$$\begin{bmatrix} \langle 00|\hat{U}|00\rangle & \langle 00|\hat{U}|01\rangle & \langle 00|\hat{U}|10\rangle & \langle 00|\hat{U}|11\rangle \\ \langle 01|\hat{U}|00\rangle & \langle 01|\hat{U}|01\rangle & \langle 01|\hat{U}|10\rangle & \langle 01|\hat{U}|11\rangle \\ \langle 10|\hat{U}|00\rangle & \langle 10|\hat{U}|01\rangle & \langle 10|\hat{U}|10\rangle & \langle 10|\hat{U}|11\rangle \\ \langle 11|\hat{U}|00\rangle & \langle 11|\hat{U}|01\rangle & \langle 11|\hat{U}|10\rangle & \langle 11|\hat{U}|11\rangle \end{bmatrix} = \begin{bmatrix} \langle 00|10\rangle & \langle 00|11\rangle & \langle 00|00\rangle & \langle 00|01\rangle \\ \langle 01|10\rangle & \langle 01|11\rangle & \langle 01|00\rangle & \langle 01|01\rangle \\ \langle 10|10\rangle & \langle 10|11\rangle & \langle 10|00\rangle & \langle 10|01\rangle \\ \langle 11|10\rangle & \langle 11|11\rangle & \langle 11|00\rangle & \langle 11|01\rangle \end{bmatrix}$$

Two qubit from 4 –qubits $\rightarrow X \otimes X \cdot X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = X \otimes I$

Changing produces an output state consisting of $|state\rangle_n$

The following circuit can be considered devoid of any gate, meaning that the output state is equal to $|0000\rangle$. The table will show the change in the states of the four qubit inputs.

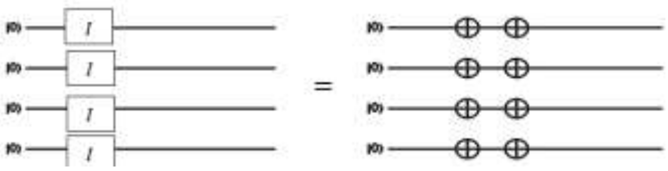


Table 2: Changing the input state $|0000\rangle$ to all four qubit states using quantum gates

	the change $|state\rangle_{1,8}$		**the change $|state\rangle_{9,16}$**
1.	the state $|0000\rangle_1$ $I \otimes I \otimes I \otimes I|q_0q_1q_2q_3\rangle$	9.	the state $|1000\rangle_9$ $X - gate\ |q_3\rangle$
2.	the state $|0001\rangle_2$ $X - gate\ |q_0\rangle$	10.	the state $|1001\rangle_{10}$ $X - gate\ |q_0\rangle, X - gate\ |q_3\rangle$
3.	the state $|0010\rangle_3$ $X - gate\ |q_1\rangle$	11.	the state $|1010\rangle_{11}$ $X - gate\ |q_1\rangle, X - gate\ |q_3\rangle$
4.	the state $|0011\rangle_4$ $X - gate\ |q_0\rangle, X - gate\ |q_1\rangle$	12.	the state $|1011\rangle_{12}$ $X - gate\ |q_0\rangle, X - gate\ |q_1\rangle$ $X - gate\ |q_3\rangle$
5.	the state $|0100\rangle_5$ $X - gate\ |q_2\rangle$	13.	the state $|1100\rangle_{13}$ $X - gate\ |q_2\rangle, X - gate\ |3\rangle$
6.	the state $|0101\rangle_6$ $X - gate\ |q_0\rangle, X - gate\ |q_2\rangle$	14.	the state $|1101\rangle_{14}$ $X - gate\ |q_0\rangle, X - gate\ |q_1\rangle$
7.	the state $|0110\rangle_7$ $X - gate\ |q_1\rangle, X - gate\ |q_2\rangle$	15.	the state $|1110\rangle_8$ $X - gate\ |q_0\rangle, X - gate\ |q_2\rangle$ $X - gate\ |q_3\rangle$
8.	the state $|0111\rangle_8$ $X - gate\ |q_0\rangle, X - gate\ |q_1\rangle$ $X - gate\ |q_2\rangle$	16.	the state $|0111\rangle_8$ $X \otimes X \otimes X \otimes X|q_0q_1q_2q_3\rangle$

Quantum computing is based on probability, the basic rule of which is superposition. The Hadamard gate is able to create superposition in any computational basis states is the basic step in setting up quantum computing [16]. That is, it performs linear superposition of all states in the logical basis [8]. This gate places the qubit in a complex superposition (linear) of its fundamental states, which, if we measure the qubit, will have equal probability of measuring 1 or 0 it would collapse to $|0\rangle$ or $|1\rangle$ [20].

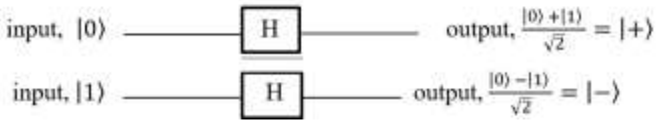
The Hadamard gate is inverted $H = H^{-1}$, $HH=I$.Dirac representation of the *H*-gate.

$$H = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)\langle 1|$$

The matrix representation for the *H*-gate

$$H = {}^{1}/_{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The basis states $|0\rangle$ and $|1\rangle$.



$$H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

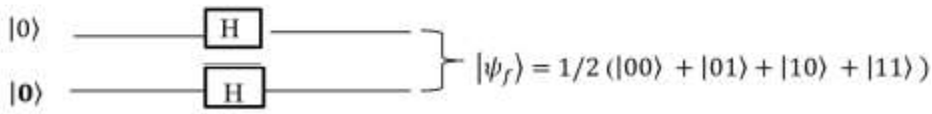
$$(H \otimes H \otimes \ldots \otimes H)\,|000\ldots0\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j\rangle$$

For the 2-qubit; $(H \otimes H)(|0\rangle \otimes |0\rangle)$

$$1/2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} |00\rangle$$

$$= 1/2 \begin{bmatrix} 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & -1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1/2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(H \otimes H)(|0\rangle \otimes |0\rangle) = 1/2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1/2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$



For the *4*-qubit.

$$H^{\otimes 4}|0\rangle^{\otimes 4} = \frac{1}{\sqrt{2^4}} \sum_{j=0}^{2^4-1} |j\rangle$$

$$\mathbf{H}^{\otimes 4}|\mathbf{0}\rangle^{\otimes 4} = (\boldsymbol{H} \otimes \boldsymbol{H} \otimes \boldsymbol{H} \otimes \boldsymbol{H})(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle)$$

When we talk about control gates that are applied to 2-qubits, we mean firstly the control, which is always represented by the identity gate, and secondly, the target, which is represented by the type of controlled-U gate.the controlled-U, or CU [20].

$$CU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & U_{21} & U_{22} \end{bmatrix}$$

He most famous and most widely used control-gate is the Control-NOT (CNOT), which we get when the target is the X-gate [20]. Dirac representation of the CNOT-gate (control the first qubit).

$$CNOT = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes \mathrm{X}$$

The matrix representation for the CNOT-gate [11].

$$CNOT = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \;\; 0] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \;\; 1] \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$CNOT = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

Note: The CNOT gate in the circuit does not interact with a qubit that is not shared with it [17].

Dirac representation of the CNOT-gate with the target being the first qubit [21].

$$CNOT = \mathbb{I} \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|$$

The matrix representation.

$$CNOT = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$$

We now observe the results of the inputs and outputs when applying the $CNOT$-gate if the first qubit is the control and the second qubit is the target [20].

Table 3: Truth table of CNOT-gate

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Let us be more comprehensive here, and mention the control-NOT gate with the state $|0\rangle$ [21], Dirac representation;

$CNOT_3 = |1\rangle\langle 1| \otimes \mathbb{I} + |0\rangle\langle 0| \otimes X$, the matrix representation.

$$CNOT_3 = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \#$$

Note: According to Qiskit's convention, For the method of keeping up with the programming system that simulates a quantum circuit, in which the highest qubit is considered the most important (little endian convention), in the third chapter we will use a calculation method in which the equation # is instead of the $CNOT_1$ gate matrix, in order to create a calculation that is easier and faster to implement [22].

To make things easier, we recall the creation of $|GHZ\rangle$ states, but this time by changing the target and controlling the *CNOT* gates on the four qubits, and the operation of the Hadamard gate on the first qubit is not constant to generate entangled states like $|GHZ\rangle$, as we will show in the following comparison.

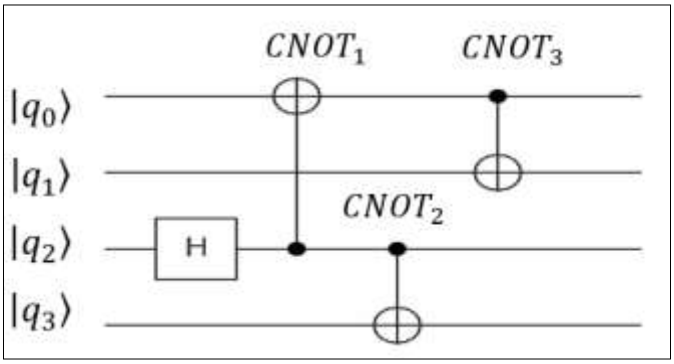


Fig 1: Two-state entanglement states of a four-qubit circuit

We will install the inputs $|0000\rangle$ on the circuit, the states will be changed using quantum gates, and in general terms it will be an X-gate. And the mathematical procedure for each change is calculated:

Input $|q_0q_1q_2q_3\rangle = |0000\rangle$, the part of the computation, The Hadamard-gate acts on the q_2 qubit.

$$|0\rangle \otimes H|0\rangle \otimes |0\rangle \otimes |0\rangle$$

$$|0\rangle \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes |00\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |00\rangle$$

The part of the computation, $CNOT_1$ gate — control $|q_2\rangle$, target $|q_0\rangle$

$$CNOT_1 = (I \otimes CNOT)(CNOT \otimes I)(I \otimes CNOT)(CNOT \otimes I)$$

$$I \otimes CNOT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad CNOT \otimes I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When we perform the usual multiplication we get:

$$CNOT_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes CNOT_1|000\rangle + |0\rangle \otimes CNOT_1|100\rangle)$$

$$CNOT_1|100\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |101\rangle, CNOT_1|000\rangle = |000\rangle$$

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |000\rangle + |0\rangle \otimes |101\rangle) = \frac{1}{\sqrt{2}}(|0000\rangle + |0101\rangle)$$

The part of the computation, $CNOT_2$ gate — control $|q_2\rangle$, target $|q_3\rangle$

$$\frac{1}{\sqrt{2}}(CNOT_2|00\rangle \otimes |00\rangle + CNOT_2|01\rangle \otimes |01\rangle)$$

$$CNOT_2|01\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle, CNOT_2|00\rangle = |00\rangle$$

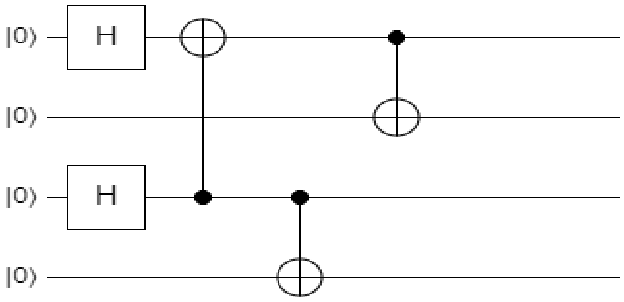
$$\frac{1}{\sqrt{2}}(|00\rangle \otimes |00\rangle + |11\rangle \otimes |01\rangle) = \frac{1}{\sqrt{2}}(|0000\rangle + |1101\rangle)$$

The part of the computation, $CNOT_3$ gate — control $|q_0\rangle$, target $|q_1\rangle$

$$\frac{1}{\sqrt{2}}(|00\rangle \otimes CNOT_3|00\rangle + |11\rangle \otimes CNOT_3|01\rangle)$$

Output: $|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0000\rangle + \frac{1}{\sqrt{2}}|1111\rangle\right)$

The two states are entangled, with a probability of 50% for each state, an amplitude equal to $1/\sqrt{2}$, and the phase angle equal to zero, considering that we did not use any gate that affects the phase.



The mathematical procedure; $|0\rangle \otimes H|0\rangle \otimes |0\rangle \otimes H|0\rangle$

$$|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

The output state of four-$state_{1,2,3,4}$; $\frac{1}{2}(|0000\rangle + |0001\rangle + |0100\rangle + |0101\rangle)$

After the set of *CNOT* gates we get the output state.

$$|\psi\rangle = \frac{1}{2}(|0000\rangle_1 + |0011\rangle_2 + |1100\rangle_3 + |1111\rangle_4)$$

Based on. 3.2, the basic format adopted in this section to compare change is.

$$\frac{1}{2}(\underbrace{|q_0q_1q_2q_3\rangle_1 + |q_0q_1q_2q_3\rangle_2} + \underbrace{|q_0q_1q_2q_3\rangle_3 + |q_0q_1q_2q_3\rangle_4})$$

q_0= 0 (Not affected by change) q_0= 1

Output has changed the two states ($state_{1,2}$)

Results and Discussion

Input is data that carries specific information that can be expressed through a specific series of values. In our field, it is expressed in binary notation {0, 1}. Output is also a specific series of values that depends on the input. The relationship between inputs and outputs depends on two stages. The first stage is the outputs in the form of an indefinite and random series in relation to the inputs, and the second stage is the final outputs for the inputs after measuring the outputs in the first stage in the required manner and in a specific order. It is impossible to establish a single relationship that generalizes to all inputs and outputs due to the different models and arrangements of quantum circuits, especially the presence of the phenomena of superposition and entanglement, which makes obtaining the outputs in a special and specific way, even if the inputs are fixed. We use the computational basis, which greatly facilitates the matter by simplifying the complexity represented by describing the states of the outputs and inputs and organizing the application of quantum gates, which are unitary gates applied in the quantum circuit. We

can find a general relationship between the inputs and outputs of the quantum circuits that we dealt with, since we dealt with four qubits and changing the states of the four qubits using quantum gates.
After using the gates g represented by *CNOT*-gates, we obtain the outputs represented by entangled states: output.

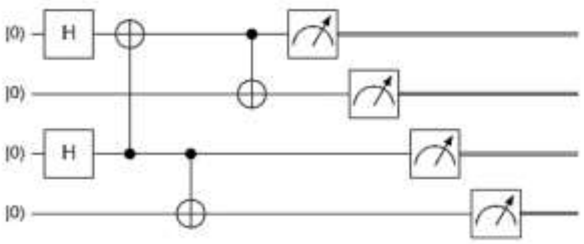
$$|\psi\rangle = (|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$$

They are the outputs of four qubits represented in this case, which we can write in the form of a column matrix consisting of 16 dimensions. We notice that according to the relationship between the inputs and the outputs that are determined by the type and arrangement of unitary operations in the quantum circuit, we obtain states that represent a different state of information, which until now before measurement is uncertain because the quantum system here is based on probability, as we agreed. By writing the states in a matrix form to illustrate the relationship between inputs and outputs in a simple way.

1	0	0	0	1	0	0	1
0	0	0	0	0	1	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0
Input				Output			

The measurement follows the purpose of using a quantum circuit. If the measurement is partial, meaning that the measurement was for the first qubit $|q_0\rangle$ and the third qubit $|q_2\rangle$, or was for the second qubit $|q_1\rangle$ and the fourth qubit $|q_4\rangle$, then possible outputs will appear that were not present in the circuit's outputs before the measurement was applied. This applies to the case of measuring one qubit $|q_i\rangle$ out of the four qubits. But if the measurement is total, meaning that all the qubits of the quantum circuit are subject to measurement, then the potential outputs, i.e. the potential measurement results, must equal one of the states of the output before the measurement. By extrapolating this here, the state of the output $|\psi\rangle$ in this section will have the following possible measurement results.

The measurement in the circuit is effective due to the presence of an H-gate that creates superposition and gives equal chance of being $|0\rangle$ and $|1\rangle$. For the qubit $|q_i\rangle$.



Steps to define a relationship
Step 1. The 4-qubit.

$$\{|q_0\rangle = |0\rangle , |q_1\rangle = |0\rangle , |q_2\rangle = |0\rangle , |q_3\rangle = |0\rangle\}$$

Input: It represents the input state of four qubits.

$$|\psi\rangle = |input\ state\rangle = |0000\rangle$$

Output: Before applying any quantum circuit and even with the presence of measurement for any n −qubit, It represents the output state $|q_0q_1q_2q_3\rangle$ of four qubits.

$$|\psi\rangle = |input\ state\rangle = |output\ state\rangle$$

Step 2. Procedure Quantum gates.
Quantum unitary gates determine the output depending on any input state, which represents the largest part of the computation.

Applying the *H*-gate to the third qubit $|q_2\rangle$ and the first qubit $|q_0\rangle$.

$$H|q_2 = 0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle, H|q_0 = 0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

Three states from The four output state contains a qubit state $|1\rangle$, Part of the qubit state of the output state.

$\{|q_3\rangle_2 = |q_1\rangle_3 = |q_1\rangle_4 = |q_3\rangle_4 = |1\rangle\}$

Applying the *CNOT*-gates

$CNOT_1$ gate; does not affect the state output in terms of calculation, but it definitely affects the relationship between the inputs and the outputs because the state output that we obtained when applying the *H* gates are inputs to the *CNOT* gates. That is, the quantum circuit considers it to exist even if nothing of the calculation is changed. The evidence for our words is that if the targeting of the $H_{|q_0\rangle}$ gate on the second qubit $|q_1\rangle$ or the third qubit $|q_4\rangle$ is changed, the $CNOT_1$ gate affects the calculation. and If the $H_{|q_0\rangle}$ gate is moved after the *CNOT* gates, we notice that the $CNOT_1$ gate will also affect the calculation and maintain on the entanglement of the output state.

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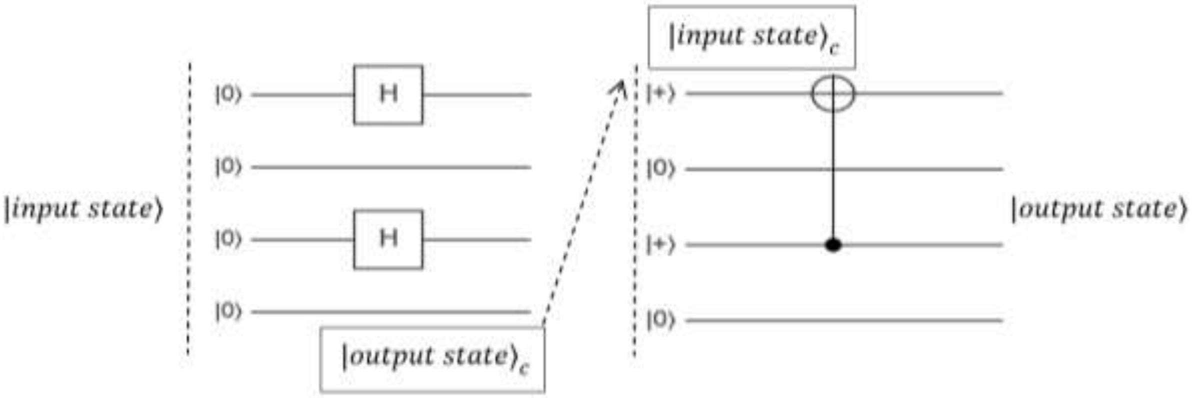
The following figure will show that the $|input\ state\rangle$ may change to a second $|input\ state\rangle_c$, or it can be considered a first $|output\ state\rangle_c$ for the same quantum circuit.



The outputs of the *H*-gates are the inputs to the $CNOT_1$ gate, so the input became a quantum circuit, the 4-qubit.

$\{|q_0\rangle = |+\rangle, |q_1\rangle = |0\rangle, |q_2\rangle = |+\rangle, |q_3\rangle = |0\rangle\}$

The input; $|\psi\rangle = |input\ state\rangle = |+0+0\rangle$

$$|\psi\rangle = |+\rangle \otimes |0\rangle \otimes |+\rangle \otimes |0\rangle\ |\psi\rangle = |+\rangle \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\right) \dashrightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes \begin{bmatrix}1\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}\right)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes \left(\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}\right)\right) = \frac{1}{\sqrt{2}}(|+\rangle \otimes (|000\rangle + |010\rangle))$$

Arrangement of qubits according to standard Z-basis and x-basis in computational basis.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\,|000\rangle \otimes |+\rangle + |010\rangle \otimes |+\rangle)$$

$$|\psi\rangle = \frac{1}{2}\,(\,|000\rangle \otimes (\,|0\rangle + \,|1\rangle) + |010\rangle \otimes (\,|0\rangle + \,|1\rangle)$$

$$|\psi\rangle = \left(\frac{1}{2}\,|0000\rangle + \frac{1}{2}|0001\rangle + \frac{1}{2}|0100\rangle + \frac{1}{2}|0101\rangle\right)$$

All this detail is for the inputs of the $CNOT_1$ gate, and the states will remain the same, but the matter is that the quantum circuit reads the $CNOT_1$ gate. We will use a different form of the $CNOT_1$ gate matrix than what we used in the methodology for calculating the $CNOT_1$ gate in this chapter, which is the traditional form of the *CNOT* gate that acts on three qubits in the reverse order of the qubits.

$$CNOT_1 - gate|\psi\rangle = \frac{1}{2}\,(\,|0\rangle \otimes (I \otimes I \otimes |0\rangle\langle 0| + X \otimes I \otimes |1\rangle\langle 1|)|000\rangle +$$

$$|0\rangle \otimes (I \otimes I \otimes |0\rangle\langle 0| + X \otimes I \otimes |1\rangle\langle 1|)|001\rangle +$$

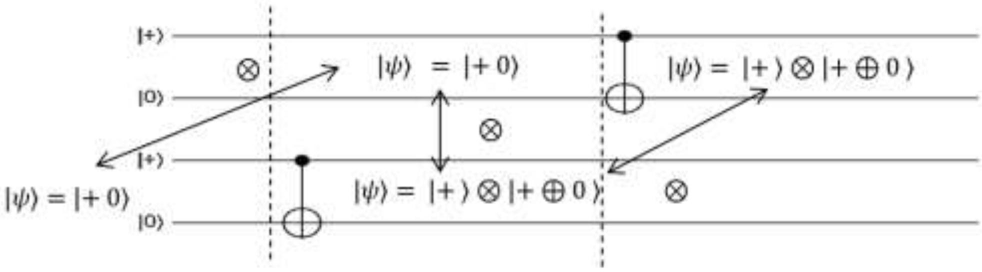
$$|0\rangle \otimes (I \otimes I \otimes |0\rangle\langle 0| + X \otimes I \otimes |1\rangle\langle 1|)|100\rangle +$$

$$|0\rangle \otimes (I \otimes I \otimes |0\rangle\langle 0| + X \otimes I \otimes |1\rangle\langle 1|)|101\rangle\,)$$

We will get output as input after computing the $CNOT_1$ gate.

$$|\psi\rangle = |output\ state\rangle = |+0+0\rangle$$

$CNOT_2$ gate and $CNOT_3$ gate; If we divide the quantum circuit into two qubits with inputs that have undergone unitary transformations that differ from the inputs of the primary quantum circuit, the $CNOT_2$ gate will operate on the third and fourth qubits $|q_2 q_3\rangle$, which gives outputs that represent inputs to the $CNOT_3$ gate for the 4-qubit quantum circuit.



The input for $CNOT_2$ gate is: $|\psi\rangle = |input\ state\rangle = |+0\rangle$

$$CNOT_2 - gate|\psi\rangle \;=\; \frac{1}{\sqrt{2}}\,(CNOT_2|0\rangle \otimes |0\rangle + CNOT_2|0\rangle \otimes |1\rangle)$$

$$CNOT_2|00\rangle = 1 \times |00\rangle + 0 \times |01\rangle + 0 \times |10\rangle + 0 \times |11\rangle$$

$$CNOT_2|01\rangle = 0 \times |00\rangle + 0 \times |01\rangle + 0 \times |10\rangle + 1 \times |11\rangle$$

$$|\psi\rangle \;= \left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \rightarrow \text{" Bell states}|\Phi^+\rangle\text{ "}$$

In the same case for the $CNOT_3$ gate, which is on the first qubit and the second qubit, considering that they are the same inputs and the same gate is applied to them, the outputs will be similar.

$$|\psi\rangle = |output\ state\rangle = |\Phi^+\rangle \;\otimes\; |\Phi^+\rangle$$

We note that any input of a qubit state $|q_i\rangle$ in a quantum circuit is linked to a relationship with the output of a gate that operates on another qubit by $\{\otimes\}$, Since we dealt with four qubits, the relationship between the inputs and outputs must contain a tensor product; $\{\otimes_{i=1}^{N}, 4-qubit: N = 4-1\}$. What helps us establish a relationship between inputs and outputs is the fact that the number of input qubits is always equal to the number of output qubits, regardless of whether the output state of the quantum circuit consists of several $|states\rangle$. Each state must contain four qubits $|q_0q_1q_2q_3\rangle$. As a general formula, by defining the Euclidean Hilbert space ($\mathcal{H}$), which allows the formation of infinite dimensional spaces, it is possible to establish a relationship between the inputs and outputs of a quantum circuit consisting of four qubits, if we consider that all gates in the circuit are one gate by defining quantum gates as a unitary operator that can be treated as a matrix. As we said earlier, we will need a 16×16 matrix because the input type is the states of four qubits and it is a 16-dimensional column matrix, and this is what linear algebra allows for matrices.

That is, the relationship between the inputs and the outputs must be determined by the gates. Even if one of the four qubits does not have a gate applied to it, the relationship between the output and the input will inevitably have an $I-gate$ applied to it. The general formula for the matrix representation of the inputs and outputs of four qubit states is as follows.

Starting from the general formula for one qubit state, we will represent four qubits through it.

- **The first qubit:** $|q_0\rangle = \sum_{i=0}^{2-1} a_{0i}|i\rangle_0 = a_{00}|0\rangle_0 + a_{01}|1\rangle_0 = \begin{bmatrix} a_{00} \\ a_{01} \end{bmatrix}$
- **The second qubit:** $|q_1\rangle = \sum_{j=0}^{2-1} a_{1j}|j\rangle_1 = a_{10}|0\rangle_1 + a_{11}|1\rangle_1 = \begin{bmatrix} a_{10} \\ a_{11} \end{bmatrix}$
- **The third qubit:** $|q_2\rangle = \sum_{u=0}^{2-1} a_{2u}|u\rangle_2 = a_{20}|0\rangle_2 + a_{21}|1\rangle_2 = \begin{bmatrix} a_{20} \\ a_{21} \end{bmatrix}$
- **The fourth qubit:** $|q_3\rangle = \sum_{v=0}^{2-1} a_{3v}|v\rangle_3 = a_{30}|0\rangle_3 + a_{31}|1\rangle_3 = \begin{bmatrix} a_{30} \\ a_{31} \end{bmatrix}$

Inputs of four qubit states

$$|\psi_i\rangle = |q_0\rangle \otimes |q_1\rangle \otimes |q_2\rangle \otimes |q_3\rangle \rightarrow \text{" input"}$$

$$|\psi_i\rangle = \sum_{i=0}^{2-1}\sum_{j=0}^{2-1}\sum_{u=0}^{2-1}\sum_{v=0}^{2-1} a_{0i}a_{1j}a_{2u}a_{3v}(|i\rangle_0 \otimes |j\rangle_1 \otimes |u\rangle_2 \otimes |v\rangle_3)$$

The operator $(\widehat{U}_{g_4})$ a unitary matrix representing a gate that operates on four qubits.

$$U_g = \prod_{i=1}^{4} g_i \otimes_{4-i}$$

$$\{U_g = gate_1 \otimes gate_2 \otimes gate_3 \otimes gate_4\}$$

$$U_g|\psi_i\rangle = |\psi_f\rangle \rightarrow \text{" output"}$$

If we multiply both sides by U_g^{-1}, we get the value of $|\psi_i\rangle$ but with a different amplitude.

$$U_g|\psi_i\rangle = gate_1 \otimes gate_2 \otimes gate_3 \otimes gate_4 \begin{bmatrix} a_{00}\begin{bmatrix} a_{10} \\ a_{11} \end{bmatrix} \\ a_{01}\begin{bmatrix} a_{10} \\ a_{11} \end{bmatrix} \end{bmatrix} \otimes \begin{bmatrix} a_{20} \\ a_{21} \end{bmatrix} \otimes \begin{bmatrix} a_{30} \\ a_{31} \end{bmatrix}$$

$$U_g(|\psi_i\rangle) = \begin{pmatrix} U_{00} & \cdots & U_{03} \\ \vdots & \ddots & \vdots \\ U_{30} & \cdots & U_{33} \end{pmatrix}_{1,2} \otimes gate_3 \otimes gate_4 \begin{bmatrix} a_{00}a_{10} \\ a_{00}a_{11} \\ a_{01}a_{10} \\ a_{01}a_{11} \end{bmatrix} \otimes \begin{bmatrix} a_{20} \\ a_{21} \end{bmatrix} \otimes \begin{bmatrix} a_{30} \\ a_{31} \end{bmatrix}$$

$$= \begin{pmatrix} U_{00} & \cdots & U_{03} \\ \vdots & \ddots & \vdots \\ U_{30} & \cdots & U_{33} \end{pmatrix}_{1,2} \otimes \begin{pmatrix} U_{00} & \cdots & U_{03} \\ \vdots & \ddots & \vdots \\ U_{30} & \cdots & U_{33} \end{pmatrix}_{3,4} \begin{bmatrix} a_{00}a_{10}\begin{bmatrix} a_{20} \\ a_{21} \end{bmatrix} \\ a_{00}a_{11}\begin{bmatrix} a_{20} \\ a_{21} \end{bmatrix} \\ a_{01}a_{10}\begin{bmatrix} a_{20} \\ a_{21} \end{bmatrix} \\ a_{01}a_{11}\begin{bmatrix} a_{20} \\ a_{21} \end{bmatrix} \end{bmatrix} \otimes \begin{bmatrix} a_{30} \\ a_{31} \end{bmatrix}$$

$\langle 0000|U_{00}|0000\rangle$

$|output\rangle$

$$= \langle Input| \begin{pmatrix} U_{00} & \cdots & \cdots & \cdots & \cdots & U_{015} \\ \vdots & \ddots & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & U_{77} & & \vdots \\ \vdots & & & & \ddots & \vdots \\ U_{150} & \cdots & \cdots & \cdots & \cdots & U_{1515} \end{pmatrix} \begin{pmatrix} a_{00}a_{10}a_{20}a_{30} \\ a_{00}a_{10}a_{20}a_{31} \\ a_{00}a_{10}a_{21}a_{30} \\ a_{00}a_{10}a_{21}a_{31} \\ a_{00}a_{11}a_{20}a_{30} \\ a_{00}a_{11}a_{20}a_{31} \\ a_{00}a_{11}a_{21}a_{30} \\ a_{00}a_{11}a_{21}a_{31} \\ a_{01}a_{10}a_{20}a_{30} \\ a_{01}a_{10}a_{20}a_{31} \\ a_{01}a_{10}a_{21}a_{30} \\ a_{01}a_{10}a_{21}a_{31} \\ a_{01}a_{11}a_{20}a_{30} \\ a_{01}a_{11}a_{20}a_{31} \\ a_{01}a_{11}a_{21}a_{30} \\ a_{01}a_{11}a_{21}a_{31} \end{pmatrix} \begin{matrix} |0000\rangle \\ |0001\rangle \\ |0010\rangle \\ |0011\rangle \\ |0100\rangle \\ |0101\rangle \\ |0110\rangle \\ |0111\rangle \\ |1000\rangle \\ |1001\rangle \\ |1010\rangle \\ |1011\rangle \\ |1100\rangle \\ |1101\rangle \\ |1110\rangle \\ |1111\rangle \end{matrix} = |output\ state\rangle \quad 3.19$$

Each state consisted of elements $a_{i,j,u,v}$ of four qubits, forming 16 states.

$|input\ state\rangle$

What determines the relationship between inputs and outputs is the intersection between the arrays of elements, which results in changing the coordinates in a tensor product, and here is the interaction that occurs for the inputs in the quantum circuit, which results in outputs. It is logical to say that the quantum circuit gives all states of the four qubits with values that are 0 or 1, and what determines this is the presence of gates, which makes every quantum circuit have a relationship between the inputs and the outputs.

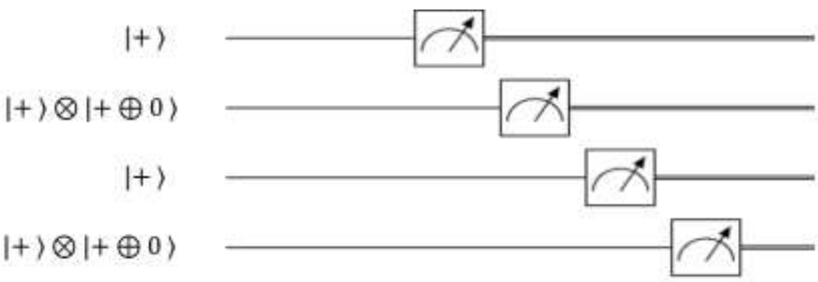
$$|\psi_f\rangle = |output\ state\rangle$$

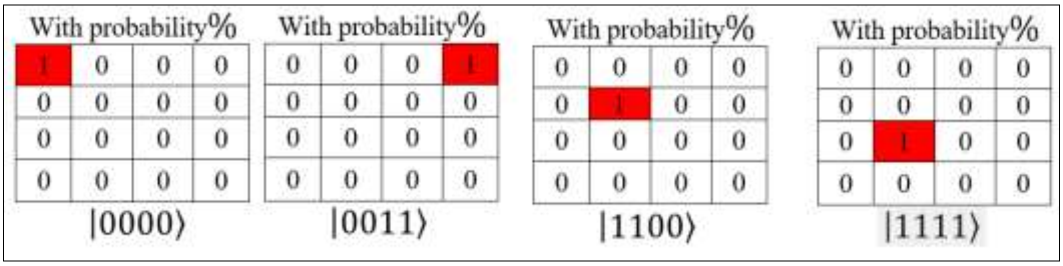
$$= 1 \times |0000\rangle + 1 \times |0011\rangle + 1 \times |1100\rangle + 1 \times |1111\rangle \quad 3.20$$

$$a_{00}a_{10}a_{20}a_{30} = 1,\ a_{00}a_{10}a_{21}a_{31} = 1, a_{01}a_{11}a_{20}a_{30} = 1, a_{01}a_{11}a_{21}a_{31} = 1$$

$$U_{00} = 1\ U_{33} = 1\ U_{1212} = 1\ U_{1515} = 1$$

Step 3. Measurement operation.
We talked about the measurement in the quantum circuit, which determines the final output of the quantum circuit by using projective procedures as an expected value $\langle\psi_f|P_{|q_i\rangle}|\psi_f\rangle, \{ i = 0,1,2,3\}$, and what we get is the result of the probability$\%$ only, since in repeating the measurement again and in the same way in which the qubits are measured, we get a different probability$\%$ and with a state vector that has an amplitude equal to 1. The probability includes all $|states\rangle$ composing the $|output\ state\rangle$.





We note that the $|input\ state\rangle = |0000\rangle$ was among the outputs, and this is not a condition in the relationship between inputs and outputs, as the $|output\ state\rangle$ may not include the $|input\ state\rangle$, and this depends on the quantum gates in the quantum circuit.

Conclusion

Quantum circuits must be built whose outputs are entangled states in order for the change to occur more precisely and for all states to include the qubits that give results that serve and benefit Quantum information science (QIS).To change the states of any quantum circuit consisting of n-qubits, you must know the quantum system in whatever field it is used, because the change gives results depending on what is present in the quantum circuit system as well as on the type of inputs. For example, teleportation requires an arrangement such that its outputs are entangled and the change is coordinated with Measurement tools, and thus change requires taking into account the quantum circuit methodology.

One of what determines the relationship between the inputs and outputs of a quantum circuit is the number of qubits in the quantum circuit, as the larger the number of qubits, the larger the *H*ilbert spaces will be as well, and this is what changes the form of inputs and outputs, even at the simple quantum level, from the classical forms of inputs and outputs.Since the inputs represent the states of four qubits $|q_0q_1q_2q_3\rangle$, meaning that the Hilbert space ($\mathfrak{H}$) will always be a 16-dimensional column matrix, which represents one state $|\text{inputs state}\rangle$ of the computational basis for the states, the outputs will have the same dimensions as the inputs, but this column matrix may contain more than the computational basis for the states $|\text{states}\rangle$.That is, the $|\text{output state}\rangle$ will inevitably represent four qubit states $|q_0q_1q_2q_3\rangle$.

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